Arbitrary Propositional Network Announcement Logic

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Outline

An Extension of propositional network announcement logic PNAL

- Reasoning about information interaction in social networks [Seligman et al., 2013, Xiong et al., 2017, Baltag et al., 2019, Morrison and Naumov, 2020];
- Quantifying over informational events [Balbiani et al., 2007, Ågotnes et al., 2010]: Add a GAL-style modality $\langle a \rangle$ for each agent a to a minimal logic for reasoning about "tweeting" (the act of making network announcements), like *Twitter*, Weibo[1]:
  - the sending by one agent of a message which received simultaneously by a number of other agents (the sender's *followers*), determined by the network structure.

[1] A Twitter-like social media application in China

Language of PNAL [Xiong et al., 2017]

Let Agnt and Prop be non-empty sets of agent names and atomic propositional letters, respectively.

Definition (Language $\mathcal{L}_{\mathsf{PNAL}}$)

The language of propositional network announcement logic (PNAL) is defined by the following grammar, where $p \in$ Prop and $a \in$ Agnt:

$$\theta \ ::= \ p \mid \neg\theta \mid \theta \wedge \theta \qquad \varphi \ ::= \ B_a\theta \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle a : \theta \rangle \varphi.$$

Models

Definition (Models)

A propositional network announcement model over Agnt and Prop is a pair (F, ω), where

- the following relation F is a binary relation on Agnt and
- the belief state function $\omega\colon \mathsf{Agnt} \to \mathrm{pow}(\mathsf{Val})$ assigns each agent a (possibly empty) set of valuations.

We write Fa for the set $\{b \mid bFa\}$ of followers of a.

Beliefs in restriction

Same as in PNAL, we restrict the beliefs of the agents, and the messages they can tweet,

- to be about propositional sentences only.

Belief update

Definition (Belief update)

When (only) agent a's belief state is updated with θ, the result is the belief state function $[a{\uparrow}\theta]\omega$. More generally, the result of updating all the agents in a set C of agents with θ is $[C{\uparrow}\theta]\omega$, where

$$[C{\uparrow}\theta]\omega(b) = \begin{cases} \omega(b) \cap [\![\theta]\!] & \text{if } b \in C \\ \omega(b) & \text{otherwise} \end{cases}$$

Satisfaction

Definition (Satisfaction)

$F, \omega \models B_a \theta$	iff	$\omega(a) \subseteq [\![\theta]\!]$
$F, \omega \models \neg\varphi$	iff	$F, \omega \not\models \varphi$
$F, \omega \models \varphi \wedge \psi$	iff	$F, \omega \models \varphi$ and $F, \omega \models \psi$
$F, \omega \models \langle a : \theta \rangle \varphi$	iff	$F, \omega \models B_a \theta$ and $F, [Fa{\uparrow}\theta]\omega \models \varphi$

Interpretation

- $B_a\theta$ for agent a believes θ, formulas of the form $B_a\theta$ are called belief formulas, expressions of the type θ are sometimes called messages;
- $\langle a : \theta\rangle\varphi$ for a can tweet θ, after which φ is the case. $[a : \theta]$ for $\neg\langle a : \theta\rangle\neg$.
- $\vec{c}$ is representing a (possibly empty) sequence of tweets, a variable over expressions of the form $c_0 : \theta_0, \ldots, c_n : \theta_n$ $(n \geq 0)$, where each c_i is an agent and each $\theta_i \in \mathcal{L}_{\text{PROP}}$.
- We write $\langle\vec{c}\rangle$ for the sequence $\langle c_0 : \theta_0\rangle \ldots \langle c_n : \theta_n\rangle$, and $[\vec{c}\,]$ for $\neg\langle\vec{c}\,\rangle\neg$.

Axiomatization of PNAL

Taut	if $\vdash_0 \varphi$ then $\vdash \varphi$	MP	if $\vdash \varphi \to \psi$ and $\vdash \varphi$ then $\vdash \psi$
K_B	$\vdash B_a(\theta \to \chi) \to (B_a\theta \to B_a\chi)$	$K_:$	$\vdash [a:\theta](\varphi \to \psi) \to ([a:\theta]\varphi \to [a:\theta]\psi)$
Nec_B	if $\vdash_0 \theta$ then $\vdash B_a\theta$	$Nec_:$	if $\vdash \varphi$ then $\vdash [a:\theta]\varphi$
Sinc	$\vdash [a:\theta]\varphi \leftrightarrow (B_a\theta \to \langle a:\theta\rangle\varphi)$	Cnsv	$\vdash B_b\chi \to [a:\theta]B_b\chi$
Rat	$\vdash \langle a:\theta\rangle B_b\chi \to B_b(\theta \to \chi)$	Foll	$\vdash \langle\vec{c}\rangle(\neg B_b\chi \wedge \langle a:\chi'\rangle B_b\chi) \to [\vec{e}][a:\theta]B_b\theta$
Null	if $\vdash_0 \theta$ then $\vdash \varphi \leftrightarrow \langle a:\theta\rangle\varphi$		

Figure: Axioms and rules of PNAL. $\varphi, \psi \in \mathcal{L}_{\mathsf{PNAL}}, \theta, \theta_i, \chi, \chi' \in \mathcal{L}_{\mathsf{PROP}}$. $\vdash_0$ denotes derivability in propositional logic.

Theorem (Theorem 3 in [Xiong et al., 2017])

PNAL *is sound and strongly complete with respect to the class of all models.*

Language and semantics of APNAL

The language of APNAL is $\mathcal{L}_{\mathsf{APNAL}}$, a conservative extension of PNAL, defined as follows, where $\theta \in \mathcal{L}_{\mathsf{PROP}}$ and $a \in$ Agnt:

$$\varphi \quad ::= \quad B_a\theta \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle a : \theta \rangle \varphi \mid [a]\varphi$$

We use derived connectives as for $\mathcal{L}_{\mathsf{PNAL}}$, in addition to $\langle a \rangle \varphi$ for $\neg[a]\neg\varphi$

Satisfaction

Definition (Satisfaction)

Satisfaction of a formula φ' in a model F, ω is defined by

$$F, \omega \models [a]\varphi \text{ iff } F, \omega \models [a : \theta]\varphi \text{ for all } \theta \in \mathcal{L}_{\mathsf{PROP}}$$

in addition to the clauses for $\mathcal{L}_{\mathsf{PNAL}}$.

In other words, $[a]$ quantifies over all possible announcements a can truthfully make. We get that

$$F, \omega \models \langle a \rangle\varphi \text{ iff } F, \omega \models \langle a : \theta \rangle\varphi \text{ for some } \theta \in \mathcal{L}_{\mathsf{PROP}}$$

Validities of APNAL

The following validities follow immediately from the semantics.

Proposition

Let $a \in \text{Agnt}$*, and* $\varphi, \psi \in \mathcal{L}_{\text{APNAL}}$*. We have*

$$\models [a]\varphi \to \varphi \qquad \models [a]\varphi \to [a:\theta]\varphi$$

$$\models [a]\varphi \to \langle a\rangle\varphi \qquad \models [a]\neg\varphi \leftrightarrow \neg\langle a\rangle\varphi$$

$$\models [a](\varphi \wedge \psi) \leftrightarrow ([a]\varphi \wedge [a]\psi) \qquad \models [a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi)$$

$$\models [a][a]\varphi \leftrightarrow [a]\varphi$$

Non-validities of the two types of modalities

The following combinations are generally *not* valid:

- $\not\models [a:\theta][b]\varphi \to [b][a:\theta]\varphi$
- $\not\models [b][a:\theta]\varphi \to [a:\theta][b]\varphi$
- $\not\models [a:\theta]\langle b\rangle\varphi \to \langle b\rangle[a:\theta]\varphi$

Church-Rosser like property

The fourth combination, is a Church-Rosser like property. The formula

$$\langle b : \chi \rangle [a : \theta] \varphi \to [a : \theta] \langle b : \chi \rangle \varphi$$

is valid [Xiong et al., 2017, Pop. 11]. The fourth combination property is in fact valid:

Proposition (Mixed-CR)

Let $a, b \in A$, $\theta \in \mathcal{L}_{\mathsf{PROP}}$, and $\varphi \in \mathcal{L}_{\mathsf{APNAL}}$. We have $\models \langle b \rangle [a : \theta] \varphi \to [a : \theta] \langle b \rangle \varphi$.

Proposition (CR)

Let $a, b \in A$, and $\varphi \in \mathcal{L}_{\mathsf{APNAL}}$. We have $\models \langle a \rangle [b] \varphi \to [b] \langle a \rangle \varphi$.

McKinsey formula

It's not hard to see that the McKinsey formula,

$$[a]\langle b\rangle\varphi \to \langle b\rangle[a]\varphi$$

is not valid, observed that

$$[a : \theta]\langle b : \chi\rangle\varphi \to \langle b : \chi\rangle[a : \theta]\varphi$$

is not valid.

Non-validities involving combinations of diamonds

- $\not\models \langle a\rangle\langle b\rangle\langle a\rangle\varphi \to \langle a\rangle\langle b\rangle\varphi$
- $\not\models \langle a\rangle\langle b\rangle\langle a\rangle\varphi \to \langle b\rangle\langle a\rangle\varphi$
- $\not\models \langle a\rangle\langle b\rangle\varphi \leftrightarrow \langle b\rangle\langle a\rangle\varphi$

Non-compactness

APNAL is not compact when there are at least two different agents. Let $b \neq a$ and

$$\Delta = \{\neg B_b p, \langle a \rangle B_b p\} \cup \{B_a \theta \to B_b \theta : \theta \in \mathcal{L}_{\mathsf{PROP}}\}.$$

We can see that Δ is unsatisfiable but any finite subset is satisfiable.

Infinitary Hilbert-style proof systems

Infinitary Hilbert-style proof systems are standard technique can be found in [Renardel de Lavalette et al., 2002, Ågotnes and Walicki, 2005, Kooi, 2006, Studer, 2008].

Our technique inspired from the completeness proof for APAL [Balbiani and van Ditmarsch, 2015] and make use of Goldblatt's necessitation forms[Goldblatt, 1982].

Necessity form

Definition (Necessity form)

Necessity forms are defined inductively as follows.

- $\#$ is a necessity form.
- If $\hat{\psi}$ is a necessity form and $\varphi \in \mathcal{L}_{\text{APNAL}}$, then $\varphi \to \hat{\psi}$ is a necessity form.
- If $\hat{\psi}$ is a necessity form and $a \in \text{Agnt}, \theta \in \mathcal{L}_{\text{PROP}}$, then $[a : \theta]\hat{\psi}$ is a necessity form.

We write $\mathcal{L}_{\text{NEC}}$ to denote the set of all necessity forms. If $\hat{\varphi}$ is a necessity form and ψ a formula, then $\hat{\varphi}(\psi)$ is the formula obtained by substituting (the unique) $\#$ in $\hat{\varphi}$ with ψ.

Axiomatization of APNAL S^ω

Definition

The derivation relation $\vdash_{S^\omega}$ ($\vdash_\omega$ for simplicity), between sets of $\mathcal{L}_{\mathsf{APNAL}}$-formulas and $\mathcal{L}_{\mathsf{APNAL}}$-formulas is the smallest relation satisfying the properties in the following figure (lower part).

[a]int	$\langle a:\theta\rangle\varphi \to \langle a\rangle\varphi$
	all $\mathcal{L}_{\mathsf{APNAL}}$ instances of PNAL axiom schemas
(**Ax**)	$\vdash_\omega \varphi$ where φ is an axiom
(**DiA**)	$\{\hat{\varphi}([a:\theta]\psi) \mid \theta \in \mathcal{L}_{\mathsf{PROP}}\} \vdash_\omega \hat{\varphi}([a]\psi)$
(**MP**)	$\{\varphi, \varphi \to \psi\} \vdash_\omega \psi$
(**Na**)	$\vdash_\omega \varphi \Rightarrow\ \vdash_\omega [a]\varphi$
(**W**)	$\Gamma \vdash_\omega \varphi \Rightarrow \Gamma \cup \Delta \vdash_\omega \varphi$
(**Cut**)	$\Gamma \vdash_\omega \Delta \ \&\ \Gamma \cup \Delta \vdash_\omega \varphi \Rightarrow \Gamma \vdash_\omega \varphi$

Figure: Axioms (upper part) and definition of the infinitary derivation relation $\vdash_\omega$ over the language $\mathcal{L}_{\mathsf{APNAL}}$ (lower part). $\Gamma \vdash_\omega \Delta$ means that $\Gamma \vdash_\omega \varphi$ for each $\varphi \in \Delta$.

Why **DIA** in necessity forms?

DIA is the only infinitary derivation rule. Let us illustrate **DIA** with some examples.

C1: $\{[a:\theta]\varphi \mid \theta \in \mathcal{L}_{\text{PROP}}\} \vdash_\omega [a]\varphi$.

C2: $\{\psi \rightarrow [a:\theta]\varphi \mid \theta \in \mathcal{L}_{\text{PROP}}\} \vdash_\omega \psi \rightarrow [a]\varphi$

C3: $\{[b:\chi][a:\theta]\varphi \mid \theta \in \mathcal{L}_{\text{PROP}}\} \vdash_\omega [b:\chi][a]\varphi$.

C4: $\{[b:\chi](\psi \rightarrow [a:\theta]\varphi) \mid \theta \in \mathcal{L}_{\text{PROP}}\} \vdash_\omega [b:\chi](\psi \rightarrow [a]\varphi)$.

C5: A general case of C3:

$$\{\underbrace{[b_1:\chi_1]\cdots[a:\theta]\cdots[b_n:\chi_n]}_{\text{finite}}\varphi \mid \theta \in \mathcal{L}_{\text{PROP}}\} \vdash_\omega \underbrace{[b_1:\chi_1]\cdots[a]\cdots[b_n:\chi_n]}_{\text{only change } one \text{ modality}}\varphi.$$

Soundness of S^{ω}

Lemma

For any formula φ and set of formulas Γ, if $\Gamma \vdash_{\omega} \varphi$ then $\Gamma \models \varphi$.

The proof is by induction on the definition of the $\vdash_{\omega}$ relation.

- Base cases: **Ax**, **MP**, and **DiA**.
- Inductive cases: **Na**, **W**, and **Cut**.

Admissible rules in S^{ω}

(**Mo**)	$\Gamma \cup \{\varphi\} \vdash_{\omega} \varphi$
(**Imp**)	$\Gamma \vdash_{\omega} \varphi \to \psi \;\&\; \Gamma \vdash_{\omega} \varphi \Rightarrow \Gamma \vdash_{\omega} \psi$
(**RT**)	$\Gamma \vdash_{\omega} \varphi \to \psi \;\Rightarrow \Gamma \cup \{\varphi\} \vdash_{\omega} \psi$
(**Ns**)	$\vdash_{\omega} \varphi \Rightarrow \; \vdash_{\omega} [a : \theta]\varphi$
(**Cond**)	$\Gamma \cup \Delta \vdash_{\omega} \varphi \Rightarrow \Gamma \cup \{\psi \to \delta \mid \delta \in \Delta\} \vdash_{\omega} \psi \to \varphi$
(**DT**)	$\Gamma \cup \{\psi\} \vdash_{\omega} \varphi \Rightarrow \Gamma \vdash_{\omega} \psi \to \varphi$
(**Raa**)	$\Gamma \cup \{\varphi\} \vdash_{\omega} \bot \Rightarrow \Gamma \vdash_{\omega} \neg\varphi$
(**Con**)	$\Gamma \vdash_{\omega} \varphi \wedge \psi \Rightarrow \Gamma \vdash_{\omega} \varphi \;\&\; \Gamma \vdash_{\omega} \psi$
(**Eqv**)	$\Gamma \vdash_{\omega} \varphi \leftrightarrow \psi \Rightarrow \Gamma \vdash_{\omega} \varphi \;\Leftrightarrow\; \Gamma \vdash_{\omega} \psi$

Figure: Admissible rules in S^{ω}.

Strong completeness of S^ω

Lemma (Lindenbaum)

Let Γ be a consistent set of formulas. There exists an MCS Γ' such that $\Gamma \subseteq \Gamma'$.

Building MCS strategy:

- **DIA-form**. A formula obtained by substitution of $[a]\psi$ on a necessity form $\hat{\varphi}$, written $\hat{\varphi}([a]\psi)$, is on **DIA-form**, the formula $\beta(:\theta) = \hat{\varphi}([a:\theta]\psi)$ is called a **DIA-witness**.
- Extension strategy: We construct $\Gamma' \supseteq \Gamma$ inductively as follows: $\Gamma_0 = \Gamma$, $\Gamma' = \bigcup_{i\in\mathbb{N}} \Gamma_i$, and
  - $\Gamma_{i+1} = \Gamma_i \cup \{\psi_{i+1}\}$, if $\Gamma_i \vdash_\omega \psi_{i+1}$;
  - $\Gamma_{i+1} = \Gamma_i \cup \{\neg\psi_{i+1}\}$, if $\Gamma_i \not\vdash_\omega \psi_{i+1}$ and ψ_{i+1} does not have the **DIA**-form;
  - $\Gamma_{i+1} = \Gamma_i \cup \{\neg\psi_{i+1}, \neg\psi_{i+1}(:\theta)\}$, if $\Gamma_i \not\vdash_\omega \psi_{i+1}$ and ψ_{i+1} has the **DIA**-form, and $\psi_{i+1}(:\theta)$ is a **DIA**-witness and $\Gamma_i \not\vdash_\omega \psi_{i+1}(:\theta)$.

Strong completeness of S^ω

Lemma (Lindenbaum)

Let Γ be a consistent set of formulas. There exists an MCS Γ' such that $\Gamma \subseteq \Gamma'$.

Building MCS strategy:

- The consistency of Γ' is showed by proving the following claim:

Claim For any Γ'' and φ such that $\Gamma'' \vdash_\omega \varphi$, we have $\Gamma'' \subseteq \Gamma' \Rightarrow \varphi \in \Gamma'$.

Canonical model and property

For any MCS Γ, F_Γ and ω_Γ are defined as follows [Xiong et al., 2017]:

- $bF_\Gamma a$ iff $[\vec{c}][a : \theta]B_b\theta \in \Gamma$ for all $\vec{c}$ and θ
- $\omega_\Gamma(a) = \bigcap\{[\![\theta]\!] \mid B_a\theta \in \Gamma\}$.

We also define the following, when Γ, Γ' are MCSs:

- Let $\langle a : \theta\rangle\Gamma = \{\varphi \mid \langle a : \theta\rangle\varphi \in \Gamma\}$.
- Let $\Gamma \trianglelefteq \Gamma'$ iff $B_a\theta \in \Gamma$ and $\Gamma' = \langle a : \theta\rangle\Gamma$ for some a and θ.
- Let $\leq$ be the transitive closure of $\trianglelefteq$.

Canonical model and property

Lemma ([Xiong et al., 2017])

If Γ *is an MCS and* $\Gamma \leq \Gamma'$ *then*

1. Γ' *is also an MCS and*
2. *there is a* $\vec{c}$ *such that:*
   (a) $\Gamma' = \langle \vec{c} \rangle \Gamma$, *and*
   (b) $[\overleftarrow{c}]\varphi \in \Gamma$ *iff* $\varphi \in \Gamma'$ *for all* φ, *where* $\overleftarrow{c}$ *is the reversal of* $\vec{c}$.

Lemma ([Xiong et al., 2017])

If $\Gamma \leq \Gamma'$ *and* $B_a\theta \in \Gamma'$ *then* $[F_\Gamma a{\uparrow}\theta]\omega_{\Gamma'} = \omega_{\langle a:\theta \rangle \Gamma'}$.

Truth Lemma and Strong Completeness

Lemma (Truth Lemma)

$F_\Gamma, \omega_\Gamma \models \varphi$ iff $\varphi \in \Gamma$, for any φ and Γ.

Theorem (Strong Completeness)

For any set of formulas Γ and formula φ, if $\Gamma \models \varphi$ then $\Gamma \vdash_\omega \varphi$.

Conclusion

- Extended PNAL with "ability" operators of the form $\langle a \rangle$ quantifying over the possible tweets agent a truthfully can make.
- A sound and strongly complete infinitary Hilbert-style axiomatic system is provided.
- For non-compactness, it is not possible to obtain a strong completeness result with a finitary proof system.
- The possibility for finitary weak completeness is left for future work.
- Another obvious direction for future work is relaxing the simplifying assumptions in the framework, in particular to allow modelling of higher-order beliefs and tweets.

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